Solution of the Inverse Conduction Problem

R. C. Mehta*
Vikram Sarabhai Space Centre, Trivandrum, India

Nomenclature

b	= slab thickness
Bo	= Biot number = hb/k
h	= heat-transfer coefficient
C_p	= specific heat
k	= thermal conductivity
t	= nondimensional time = $\alpha\tau/b^2$
T	= temperature
T_g	= driving gas temperature
T_0	= initial gas temperature
x	= space coordinate
X	= x/b
α	= thermal diffusivity
ρ	= density
τ	= time
θ	= nondimensional temperature = $(T-T_0)/(T_g-T_0)$

Introduction

MANY rocket engines operate in the turbulent boundary-layer regime, and consequently there is considerable interest in calculating convective heat transfer from turbulent boundary layers in nozzles. It is realized that such calculations are essentially empirical because of our limited knowledge of the effect of acceleration on the flow and thermal structure of turbulent boundary layers and of the simultaneous effects of wall cooling and compressibility on the flow of rocket combustion gases and heated air. In heat-transfer studies, many experimental difficulties may arise in implanting heat-flux sensors or thermocouples at the surface for heat-transfer measurements. Furthermore, the presence of a probe at the surface disturbs the condition of the boundary and the flow process adjacent to it and thus actual wall heat flux. It is therefore desirable in these circumstances that the prediction of surface temperature and heat flux be accomplished by inverting the temperature as measured by a probe located interior to the surface of the solid material. Such a problem is termed the inverse problem.

Problems of the foregoing kind have been studied by several investigators over the past two decades. Stolz[1] and Beck[2] considered the numerical inversion of the integral solution for semi-infinite and spherical bodies. Other papers using least squares were written by Frank[3] and Burggraf.[4] Carslaw and Jaeger[5] and Shumakov[6] applied different series approaches in which generally the local temperature and local heat flux at an interior location and their higher derivatives are required. Sparrow et al.[7] and Imber and Khan[8] utilized the transform method. Beck[9] used a finite-difference approximation in conjunction with a least-squares fit procedure as well as a nonlinear estimate method for the inverse conduction problem. Most of these analyses assumed a one-dimensional model, but in reality the temperature field is distorted to become two- or three-dimensional when a cavity is drilled to accommodate the thermocouple leads. The degree of distortion may be influenced by the dissimilar properties of the thermocouple and surrounding material, and by the diameter and depth of the cavity.[10]

The present Note reports an iterative scheme to obtain values of surface temperature and convective heat-transfer coefficient from the measured temperature history at the outer surface of the nozzle.

Received March 24, 1977.
Index categories: Solid and Hybrid Rocket Engines; Heat Conduction.

*Engineer, Propulsion Engineering Division (PSN).

Analysis

With the usual assumptions and following nondimensionalization, the equations to be solved are

$$\partial\theta/\partial t=\partial^2\theta/\partial X^2; \quad 0<X<1, \quad t>0 \tag{1}$$

$$\partial\theta(0,t)/\partial X=Bo[\theta(0,t)-1], \quad t>0 \tag{2}$$

$$\partial\theta(1,t)/\partial X=0, \quad t>0 \tag{3}$$

$$\theta(X,0)=0 \quad \text{for all} \quad X \tag{4}$$

The constant thermal properties solution of this problem is straightforward and is given by

$$\theta(X,t)=1-2\sum_{m=1}^{\infty}\frac{Bo}{(Bo^2+\lambda_m^2+Bo)}\frac{\cos\lambda_m(1-X)}{\cos\lambda_m}\exp(-\lambda_m^2 t) \tag{5a}$$

$$\lambda\tan\lambda=Bo \tag{5b}$$

In the foregoing solution [Eqs. (5)], Bo is an unknown parameter. In estimating Bo, one minimizes

$$F(Bo)=[\theta_c(X,t)-\theta_m(X,t)] \tag{6}$$

where θ_c and θ_m are, respectively, the calculated and measured temperatures at (X,t). The Taylor series in its simplest form is usually adequate; it is outlined in the following.

The calculated temperature is, in general, a nonlinear function of Bo. The Taylor series method is an iterative procedure, however, which assumed at each step that the temperature is a linear function of Bo, or

$$\theta(Bo)\approx\theta(B\overset{n}{o})+\Delta B\overset{n+1}{o}(\partial\theta/\partial Bo) \tag{7}$$

where

$$\Delta B\overset{n+1}{o}=B\overset{n+1}{o}-B\overset{n}{o} \tag{8a}$$

$$\frac{\partial\theta}{\partial Bo}\approx\frac{\theta[B\overset{n}{o}(1+\epsilon)]-\theta(B\overset{n}{o})}{\epsilon Bo} \tag{8b}$$

The n superscript is an index related to the number of the iteration. If ϵ is made equal to 10% of Bo, $\partial\theta/\partial Bo$ is approximated accurately. The temperatures on the right-hand side of Eq. (8b) are calculated using Eqs. (5) twice with $B\overset{n}{o}$ and then with $B\overset{n+1}{o}(1+\epsilon)$. Using $\partial F/\partial Bo=0$, a correction in Bo is given by

$$\Delta B\overset{n+1}{o}=-\theta(B\overset{n}{o})/(\partial\theta/\partial Bo) \tag{9}$$

This iterative procedure begins with an estimated value of $B\overset{0}{o}$, corresponding to $n=0$, and continues for increasing values of the integer n until $|F|$ is less than, say, 10^{-4}. The method estimates the components of the heat flux one at a time and thus may be considered an on-line method.[9]

Convergence

The convergence of the iteration method is now examined. It is mentioned by Scarborough[11] that the criterion for convergence for the iteration is given by

$$|\{\theta(Bo)\cdot(\partial^2\theta/\partial B^2o)\}/(\partial\theta/\partial Bo)^2|<1 \tag{10}$$

If a small value of Bo is chosen initially, then this convergence criterion can be satisfied easily; however, should this criterion not be satisfied, then the value of Bo can be decreased until

Table 1 Comparison of present solution with Bartz's solution

τ, sec	$X=0.0$	$X=0.2$	$X=0.4$	$X=0.6$	$X=0.8$	$X=1.0$	θ_m at outer surface	h_p,[a] W/m^2-K	h_B,[b] W/m^2-K	ϵ[c]
			θ_c							
6	0.29505	0.16941	0.08610	0.03859	0.01616	0.00981	0.00983	1821.9	2254.2	19.176
7	0.31093	0.18719	0.10164	0.04987	0.02373	0.01596	0.01588	1810.0	2254.2	19.705
8	0.29965	0.18677	0.10657	0.05607	0.02941	0.02122	0.02116	1610.3	2254.2	28.562
9	0.32442	0.20891	0.12459	0.06965	0.03960	0.03016	0.03020	1690.9	2254.2	24.991
10	0.33400	0.22077	0.13650	0.08022	0.04867	0.03860	0.03855	1669.7	2254.2	25.929
11	0.34157	0.23088	0.14725	0.09033	0.05782	0.04732	0.04724	1641.9	2254.2	27.162
12	0.33017	0.22716	0.14860	0.09446	0.06315	0.05296	0.05291	1497.6	2254.2	33.563
13	0.33116	0.23173	0.15521	0.10188	0.07073	0.06052	0.06046	1443.1	2254.2	35.983
14	0.33094	0.23509	0.16080	0.10858	0.07783	0.06771	0.06764	1387.0	2254.2	38.470
15	0.34418	0.24817	0.17321	0.12008	0.08857	0.07815	0.07823	1413.0	2254.2	37.316
16	0.34751	0.25380	0.18027	0.12785	0.09661	0.08625	0.08615	1383.7	2254.2	38.617

[a] h_p = heat-transfer coefficient (present). [b] h_B = heat-transfer coefficient (Bartz). [c] ϵ = percentage error $\{(h_B-h_p)/h_B\}\times 100$.

convergence is assumed. Upon further examination of Eqs. (5), it is found that, for small values of Bo, $\partial^2\theta/\partial Bo^2 \approx (\partial\theta/\partial Bo)^2$. Thus, if a small initial value of Bo is used, the convergence criterion for rapid convergence becomes

$$|Bo| < 1.5708 \tag{11}$$

The computer technique for solving for a value of Bo for a particular location and time is as follows. Start with a small initial value of Bo, satisfy the convergence criterion, and untilize the Newton-Raphson method to solve for the correct value of Bo.

Example

Using the present method, estimation of convective heat-transfer coefficient is carried out in conjunction with the experimental data of outer surface temperatures of M.S. used for nozzle divergent in a rocket motor static test. Insulated chromel/alumel thermocouples of 30 gage were used for measuring the temperature. The nozzle conditions and material properties taken are $b=0.0211$m, $T_o=300$ K, $\rho=7900$ kgm^3, $C_p=545$ W-sec/kg$-$K, $T_g=2946.2$ K, thermal conductivity (average)$=35$ W/mK, and burning time$=16$ sec. It is seen from Table 1 that estimated values of the convective heat-transfer coefficient are somewhat lower than the calculated results of Bartz.[12] Thus, it shows that Bartz's equation gives conservative estimation for the convective heat transfer. This is also is demonstrated experimentally by Brinsmade and Desmon.[13]

Conclusions

The Newton-Raphson iteration procedure proves quite useful in estimating the value of the heat-transfer coefficient from temperature data measured on the outer surface of the nozzle. The convergence criteria for the iteration are indicated. Extension of the method for two dimensions is straightforward. It has the advantage that temperatures can be found directly at specified time and location whereas the numerical approach requires the development of the temperature profile from the initial state.

References

[1] Stolz, G., "Numerical Solutions to an Inverse Problem of Heat Conduction for Simple Shapes," *Journal of Heat Transfer*, Vol. 82, Feb. 1960, pp. 20-26.

[2] Beck, J.V., "Surface Heat Flux Determination Using an Integral Method," *Nuclear Engineering Design*, Vol. 7, 1968, pp. 170-178.

[3] Frank, I., "An Application of Least Squares Method to the Solution of the Inverse Problem of Heat Conduction," *Journal of Heat Transfer*, Vol. 85, Nov. 1963, pp. 378-379.

[4] Burggraf, O.R., "An Exact Solution of the Inverse Problem in Heat Conduction Theory and Applications," *Journal of Heat Transfer*, Vol. 86, Aug. 1964, pp. 373-382.

[5] Carslaw, H.D. and Jaeger, J.C., *Conduction of Heat in Solids*, Oxford University Press, London, 1959, p. 12.

[6] Shumakov, N.V., "A Method for the Experimental Study of the Process of Heating a Solid Body," *Journal of Technical Physics of the Academy of Sciences USSR*, Vol. 2, 1957, pp. 771-777.

[7] Sparrow, E.M., Hadji-Sheikh, A., and Lundgren, T.S., "The Inverse Problem in Transient Heat Conduction," *Journal of Applied Mechanics*, Vol. 86, Sept. 1964, pp. 369-375.

[8] Imber, M. and Khan, J., "Prediction of Transient Temperature Distributions with Embedded Thermocouples," *AIAA Journal*, Vol. 10, June 1972, pp. 784-789.

[9] Beck, J.V., "Nonlinear Estimation Applied to the Nonlinear Inverse Heat Conduction Problem," *International Journal of Heat and Mass Transfer*, Vol. 13, April 1970, pp. 703-716.

[10] Chen, C.J. and Danh, T.M., "Transient Temperature Distortion in a Slab Due to Thermocouple Cavity," *AIAA Journal*, Vol. 14, July 1976, pp. 979-981.

[11] Scarborough, J.B., *Numerical Mathematical Analysis*, Oxford and IHB Publishing Co., Calcutta, India, 1968, pp. 208-211 (Indian ed.).

[12] Bartz, D.R., "A Simple Equation for Rapid Estimation of Rocket Nozzle Convective Heat Transfer Coefficients," *Jet Propulsion*, Vol. 27, Jan. 1957, pp. 49-51.

[13] Brinsmade, A.F. and Desmon, L.G., "Hypothesis for Correlating Rocket Nozzle Throat Convective Heat Transfer," *Heat Transfer – Cleveland, Chemical Engineering Progress Symposium*, Vol. 61, No. 59, 1965, pp. 88-98.

Three-Dimensional Moisture Diffusion in Laminated Composites

J. M. Whitney*
Air Force Materials Laboratory, Wright-Patterson Air Force Base, Ohio

Introduction

THE mathematics of moisture diffusion in laminated composites are based on Fick's analogy to Fourier's heat conduction equation.[1] The application of Fick's Law to one-dimensional, through-the-thickness moisture absorption in composite laminates has been established by Shen and

Presented as Paper 77-397 at the AIAA/ASME 18th Structures, Structural Dynamics, and Materials Conference, San Diego, Calif., March 21-23, 1977; submitted March 17, 1977; revision received June 10, 1977.

Index categories: Materials, Properties of; Structural Composite Materials.

*Materials Research Engineer, Nonmetallic Materials Division. Associate Fellow AIAA.